Math 102

Krishanu Sankar

November 27, 2018

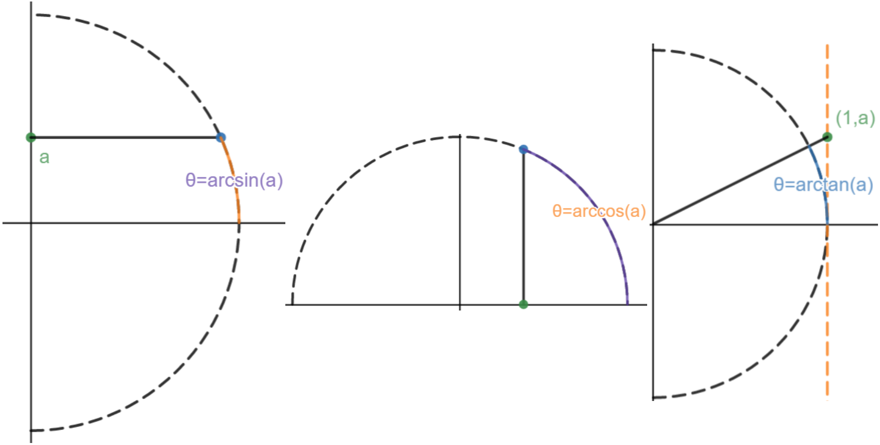
Announcements

- Review sessions on Thursday and Friday 4-7PM, in SWNG 122.
- A & D - deadline to register is today!

Goals Today

- Inverse Trigonometric Functions
  - Review
  - Derivatives
- Applications
  - Related Rates
  - Optimization

Recall - $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$



$\cos(\arcsin(x))$

Question: Calculate $\cos(\arcsin(\frac{1}{3}))$. Can you find a general expression for $\cos(\arcsin(x))$?

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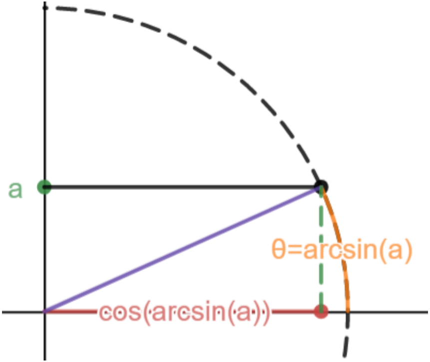

$$\boxed{\cos(\arcsin(x)) = \sqrt{1-x^2}}$$

Therefore,

$$\cos(\arcsin(1/3)) = \sqrt{1-\frac{1}{9}}$$

$$= \frac{2\sqrt{2}}{3}$$

Derivative of $\arcsin(x)$

$$y = \arcsin(x) \quad \Longrightarrow \quad \sin(y) = x$$

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$$\Longrightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$

using the fact that $\cos(\arcsin(x)) = \sqrt{1-x^2}$.

Derivative of $\arcsin(x)$ - another way

$$y = \arcsin(x) \quad \Longrightarrow \quad \sin(y) = x$$

$$\Longrightarrow \frac{d\sin(y)}{dx} = 1 \quad \Longrightarrow \quad \cos(y)\frac{dy}{dx} = 1$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$

using the fact that $\cos(\arcsin(x)) = \sqrt{1-x^2}$.

Exercise: Calculate $\frac{d}{dx}(\arccos(x))$. Start by letting $y = \arccos(x)$, converting this to $\cos(y) = x$ and then taking $\frac{d}{dy}$ of both sides.

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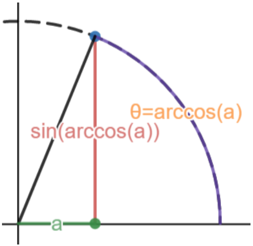
$$y = \arccos(x) \quad \Longrightarrow \quad \cos(y) = x$$

$$\Longrightarrow \quad \frac{d\cos(y)}{dy} = \frac{dx}{dy} \quad \Longrightarrow \quad -\sin(y) = \frac{dx}{dy}$$

$$\Longrightarrow \quad \frac{dy}{dx} = -\frac{1}{\sin(\arccos(x))}$$

$$= -\frac{1}{\sqrt{1-x^2}}$$



Derivative of $\arctan(x)$

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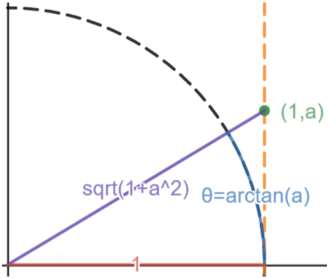
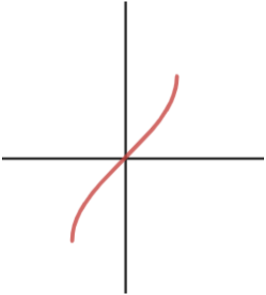
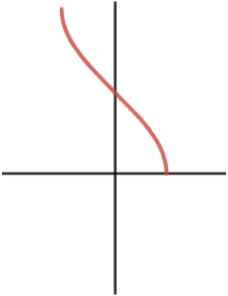
$$\implies \frac{dy}{dx} = \cos^2(y) = \cos^2(\arctan(x)) = \frac{1}{1+x^2}$$



Exercise: Sketch the derivatives!



$y = \arcsin(x)$



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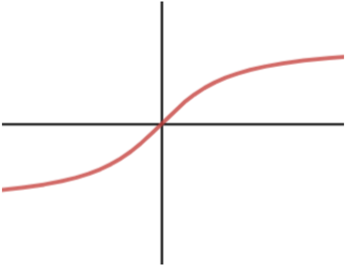

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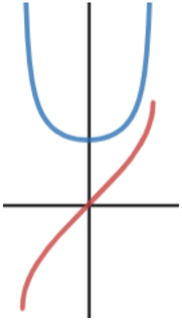
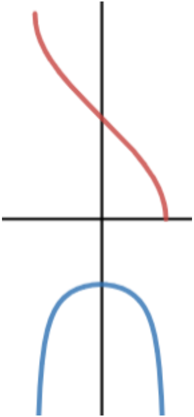
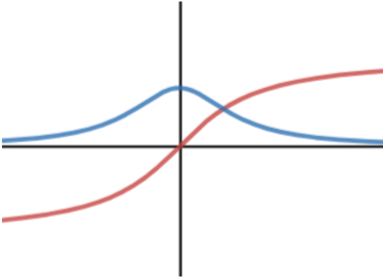


$$y = \arcsin(x)$$
$$y = \frac{1}{\sqrt{1-x^2}}$$

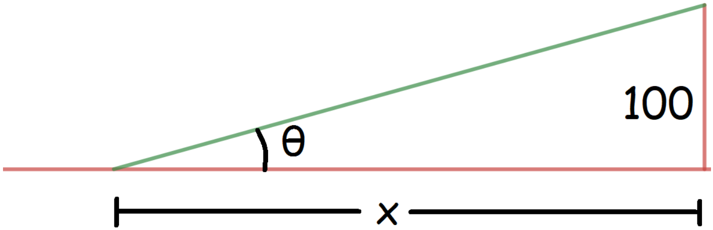


$$y = \arccos(x)$$
$$y = -\frac{1}{\sqrt{1-x^2}}$$



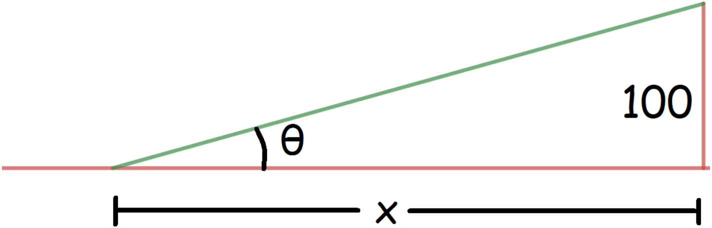
$$y = \arctan(x)$$
$$y = \frac{1}{1+x^2}$$

Viewing Angle



Question: A building is 100m tall. You walk away from it at a constant speed of 1 m/s. The *viewing angle* is the angle formed between your line of sight to the top and your line of sight to the bottom (assume for simplicity that you are 0m tall). When you are a distance of x away from the base, how quickly is the viewing angle, θ, changing?

Viewing Angle - Solution

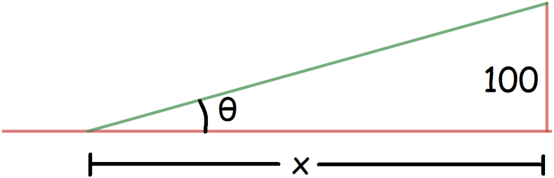


$$\tan(\theta) = 100/x \quad \implies \quad \theta = \arctan(100/x)$$

$$\implies \frac{d\theta}{dt} = \frac{1}{1 + (100/x)^2}\left(-\frac{100}{x^2}\right)\frac{dx}{dt}$$

Since $\frac{dx}{dt} = 1$, we can simplify the above expression to get $\frac{d\theta}{dt} = -\frac{100}{x^2+100^2}$.

Viewing Angle - Alternate Solution



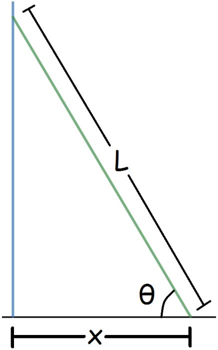
$$\tan(\theta) = 100/x \implies \frac{d\tan(\theta)}{dt} = \frac{d(100/x)}{dt}$$

$$\implies \sec^2(\theta)\frac{d\theta}{dt} = -\frac{100}{x^2}\frac{dx}{dt}$$

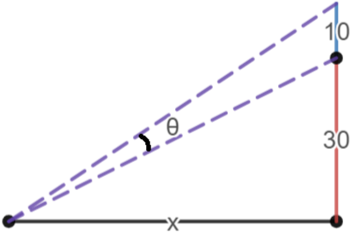
$$\implies \frac{d\theta}{dt} = -\frac{100}{x^2}\frac{dx}{dt}\cos^2(\theta)$$

You can calculate from the diagram that $\cos^2(\theta) = \frac{x^2}{x^2+100^2}$. Since $\frac{dx}{dt} = 1$, we can simplify the above expression to get $\frac{d\theta}{dt} = -\frac{100}{x^2+100^2}$.

Question: The base of a ladder of length L is pulled away from the wall at a constant rate of 0.5 m/s. In terms of x, how quickly is the angle θ changing?



Question: A billboard has height 10m and its bottom is at height 30m. At what distance x from the billboard will you get the largest viewing angle, θ?



Slipping Ladder - Solution

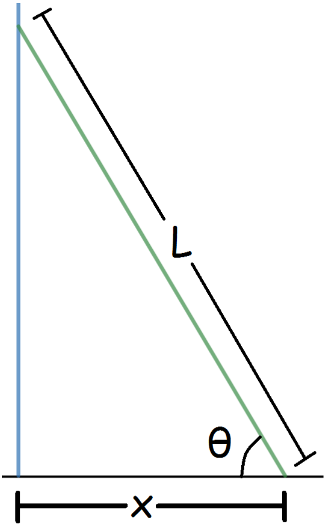
$$\theta = \arccos(\frac{x}{L})$$

$$\frac{d\theta}{dt} = \frac{d}{dt}\arccos(\frac{x}{L})$$

$$= -\frac{1}{\sqrt{1-(x/L)^2}}\frac{d(x/L)}{dt}$$

$$= -\frac{1}{L\sqrt{1-(x/L)^2}}\frac{dx}{dt}$$

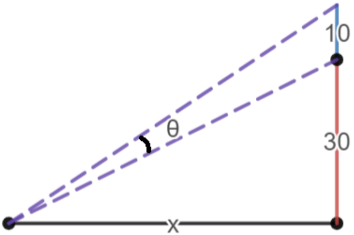
$$= -\frac{0.5}{\sqrt{L^2-x^2}}$$



Billboard - Solution

The angle θ is given by

$$\theta = \arctan\left(\frac{30}{x}\right) - \arctan\left(\frac{40}{x}\right)$$



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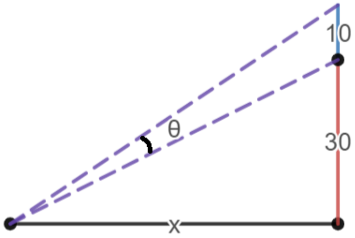

$$\frac{d\theta}{dx} = \frac{1}{1+(30/x)^2}\left(-\frac{30}{x^2}\right) - \frac{1}{1+(40/x)^2}\left(-\frac{40}{x^2}\right)$$

$$= -\frac{30}{x^2+30^2} + \frac{40}{x^2+40^2}$$

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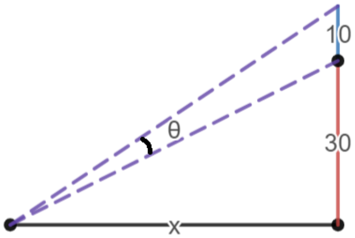

$$\theta = \arctan\left(\frac{40}{x}\right) - \arctan\left(\frac{30}{x}\right)$$

$$\frac{d\theta}{dx} = -\frac{40}{x^2+40^2} + \frac{30}{x^2+30^2}$$

Set this equal to zero to find the critical point.

$\frac{30}{x^2+30^2} = \frac{40}{x^2+40^2}$

$$\implies 30x^2 + 30(40^2) = 40x^2 + 40(30^2)$$

$$\implies x = \sqrt{1200} = 20\sqrt{3}$$